Linear Algebra II

08/04/2013, Monday, 9:00-12:00

The exam consists of 5 problems. At each problem, the points for each subproblem are indicated (10 pts gratis). You are **NOT** allowed to use **any kind of calculators**. Good luck!

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1 Inner product spaces and Gram-Schmidt process $3 + 6 + (6 + 3) = 18$ pts

(a) Let v_1, v_2, and v_3 be nonzero vectors of an inner product space V. Show that v_1, v_2, and v_3 are linearly independent if $\{v_1, v_2, v_3\}$ is an orthogonal set of vectors.

(b) Consider the vector space P_4. Let

$$\langle p, q\rangle = p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3$$

where $p(x) = p_0 + p_1x + p_2x^2 + p_3x^3$ and $q(x) = q_0 + q_1x + q_2x^2 + q_3x^3$. Show that $\langle \cdot, \cdot\rangle$ is an inner product on P_4.

(c) Consider the vector space P_4. Let S be the subspace spanned by the vectors 1, $1+x$, and $(1+x)^2$.

(i) By applying the Gram-Schmidt process, find an orthonormal basis for the subspace S.

(ii) Find the vector p in S that is closest to the vector $(1+x)^3$.

2 Eigenvalues and diagonalization $4 + 4 + (2 + 8) = 18$ pts

(a) Let $M \in \mathbb{C}^{m\times m}$ be of the form $M = A + iB$ where A, $B \in \mathbb{R}^{m\times m}$. Show that M is Hermitian if and only if $A = A^T$ and $B = -B^T$.

(b) Let $M \in \mathbb{R}^{m\times m}$ be a matrix with $M = \alpha M^T$ for some real number α. Show that M is unitarily diagonalizable.

(c) Consider the matrix

$$M = \begin{bmatrix} 3 & -2 & 2 \\ -2 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}.$$

(i) Show that the eigenvalues are -1, -1 and 5.

(ii) Find an orthogonal matrix U that diagonalizes M.

3 Positive definiteness $6 + 12 = 18$ pts

(a) Let $M \in \mathbb{R}^{m\times m}$ be a symmetric matrix. Show that M^2 is positive definite if and only if M is nonsingular.

(b) Check if the matrix

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

is positive definite or not.

4 Singular value decomposition $13 + 5 = 18$ pts

Consider the matrix

$$M = \begin{bmatrix} a & -b & 0 \\ b & a & -b \\ 0 & b & a \end{bmatrix}$$

where a and b are positive real numbers.

(a) Find a singular value decomposition of M.

(b) Find the best rank 2 approximation of M.

5 Cayley-Hamilton theorem and Jordan canonical form $6 + 12 = 18$ pts

Consider the matrix

$$M = \begin{bmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}$$

where a and b are real numbers with $a \neq 0$.

(a) By using Cayley-Hamilton theorem, show that $(M^2 - I)^{999} = 0$.

(b) Put the matrix M into Jordan canonical form.